Christ Church
Grammar School

2021
TEST 1

MATHEMATICS SPECIALIST Year 12

Calculator-free

Your name SOLUTIONS

Teacher's name ______________________________

Time and marks available

Reading Time: 4 minutes
Working time for this section: 40 minutes
Marks available: 42 marks

Materials required/recommended

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet using a blue/black pen. Do not use erasable/gel pens.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

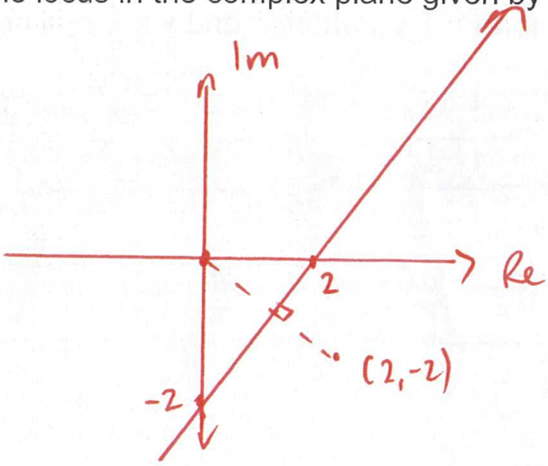
6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 (6 marks)

Given $z = x + yi$, sketch the locus in the complex plane given by

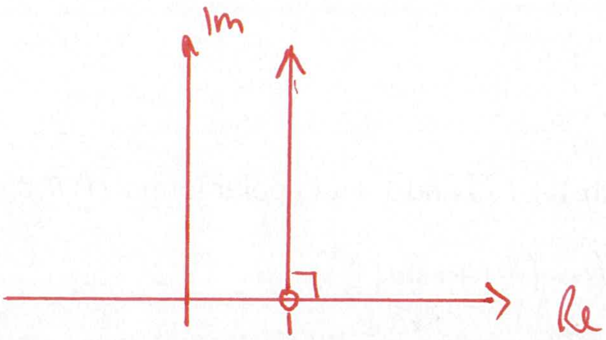
(a) $|z| = |z - 2 + 2i|$. (2 marks)



✓ straight line; ⊥ bisecting (0,0) + (2,-2)

✓ indicates 'x' + 'y' intercepts.

(b) $\mathrm{Arg}\,(z - 1) = \frac{\pi}{2}$ (2 marks)



✓ vertical line at Re is 1.

✓ open circle at 1

(c) Describe the locus of w if z describes a circle with centre $1 + 2i$ and radius of 3, when $w = 2z$. (2 marks)

Circle with centre at $2 + 4i$ ✓ states circle and correct new centre

radius of 6. ✓ states correct new radius.

Question 2 **(7 marks)**

(a) Write $\frac{1+i\sqrt{3}}{1+i}$ in the form $x+yi$, where x and y are real numbers. (2 marks)

$$\frac{(1+i\sqrt{3})}{(1+i)} \times \frac{(1-i)}{(1-i)}$$ ✓ multiplies by conjugate

$$= \frac{1 - i + i\sqrt{3} + \sqrt{3}}{2}$$

$$= \frac{1+\sqrt{3}}{2} + \left(\frac{\sqrt{3}-1}{2}\right)i$$ ✓ obtains correct answer.

(b) By expressing both $1+i\sqrt{3}$ and $1+i$ in polar form $r \text{ cis } \theta$, show that

$$\frac{1+i\sqrt{3}}{1+i} = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right).$$ (3 marks)

$1+i\sqrt{3} = 2 \text{ cis } \pi/3$ ✓ correct polar form

$1+i = \sqrt{2} \text{ cis } \pi/4$ ✓ correct polar form

$$\frac{1+i\sqrt{3}}{1+i} = \frac{2 \text{ cis } \pi/3}{\sqrt{2} \text{ cis } \pi/4}$$

$$= \sqrt{2} \text{ cis}(\pi/3 - \pi/4)$$ ✓ correct working shown.

$$= \sqrt{2} \text{ cis } \pi/12$$

(c) Hence, using your answers from parts (a) and (b), determine the exact value of $\sin\left(\frac{\pi}{12}\right)$. (2 marks)

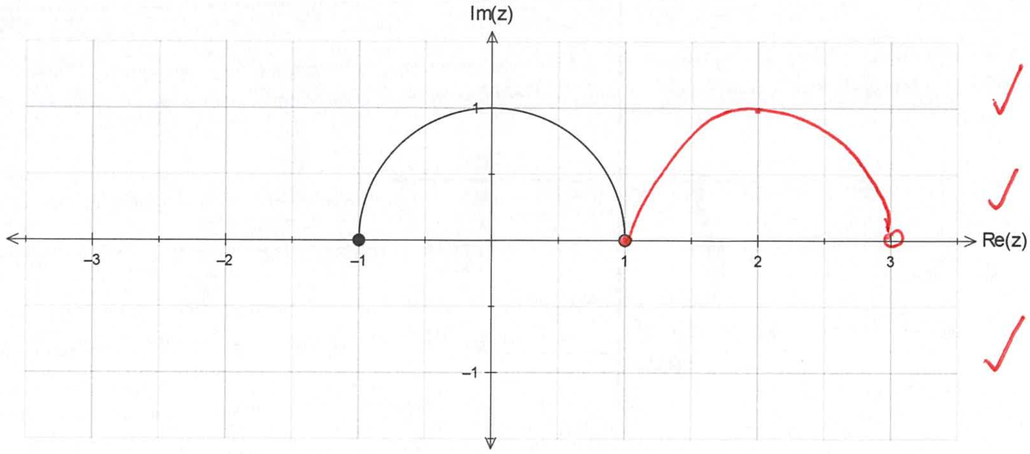
$$\sqrt{2}\sin\pi/12 = \frac{\sqrt{3}-1}{2}$$ ✓ equate imaginary parts

$$\therefore \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$ ✓ correct simplification

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Question 3 **(5 marks)**

The diagram shows the locus of all points that satisfy the conditions $|z| = 1$ and $0 < \arg(z) \leq \pi$.



✓ semi-circle (as close as possible)

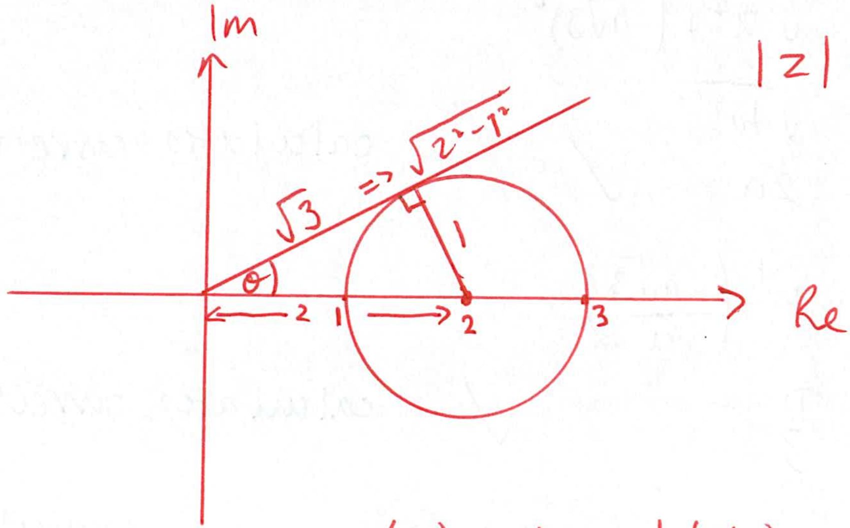
✓ centre at (2,0) radius 1

✓ closed circle at $x = 1$, open circle at $x = 3$.

(a) Sketch on the same axes the locus of all points defined by $|z - 2| = 1$ and $0 < \arg(z - 2) \leq \pi$. (3 marks)

see diagram

(b) Determine the maximum value of $\arg(z)$ for your answer in (a), and the exact value of $|z|$ at this point. (2 marks)



$|z| = \sqrt{3}$.

$\arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

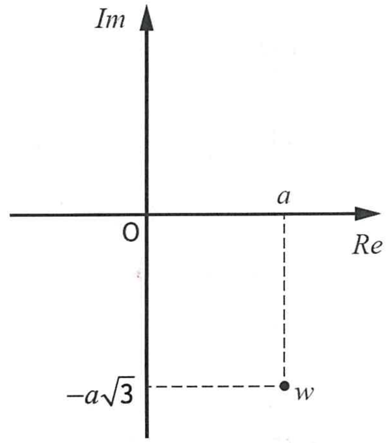
$= \frac{\pi}{6}$

$\therefore$ max $\arg(z) = \frac{\pi}{6}$ and $|z| = \sqrt{3}$. ✓ calculates correct magnitude

✓ calculates correct max angle

Question 4 **(10 marks)**

The complex number w has been plotted on an Argand diagram, as shown below.



(a) Express w in

(i) Cartesian form. (1 mark)

$a - a\sqrt{3}\,i$ ✓

(ii) Polar form. (3 marks)

$$|w| = \sqrt{a^2 + (-a\sqrt{3})^2}$$
$$= \sqrt{4a^2}$$
$$= 2a \quad ✓$$

calculates correct modulus

$$\theta = \tan^{-1}\left(\frac{-a\sqrt{3}}{a}\right)$$
$$= -\frac{\pi}{3} \quad ✓$$

calculates correct angle

$$\therefore\ w = 2a\,\mathrm{cis}\left(-\pi/3\right) \quad ✓$$

explicitly writes in polar form.

or $w = 2a\left(\cos\left(-\pi/3\right) + i\sin\left(-\pi/3\right)\right)$.

Question 4 continued

(b) The complex number z_1 is a root of $z^3 = w$, where

$$z_1 = k\left(\cos\left(\frac{\pi}{m}\right) + i\sin\left(\frac{\pi}{m}\right)\right) \qquad \text{for integers } k \text{ and } m.$$

Given that $a = 4$,

(i) Use de Moivre's theorem to obtain the values of k and m. (4 marks)

$W = 8 \text{ cis}(-\pi/3) = z^3$ ✓ states 'w' with a = 4.

$z_1 = 8^{1/3} \text{ cis}(-\pi/3)^{1/3}$

$= 2 \text{ cis}(-\pi/9)$ ✓ shows use of de Moivre's theorem

$\therefore k = 2$ ✓ states 'k' value explicitly

$m = -9$ ✓ states 'm' value explicitly.

(ii) determine the remaining roots. (2 marks)

3 roots $\therefore$ each $2\pi/3$ apart.

$z_1 = 2\text{cis}(-\pi/9)$

$z_2 = 2\text{ cis}\left(-\frac{\pi}{9} + \frac{2\pi}{3}\right)$

$= 2\text{ cis}\left(\frac{5\pi}{9}\right)$ ✓ correct 2^{nd} root (no working required)

$z_3 = 2\text{ cis}\left(\frac{5\pi}{9} + \frac{2\pi}{3}\right)$

$= 2\text{ cis}\left(\frac{11\pi}{9}\right)$

$= 2\text{ cis}\left(-\frac{7\pi}{9}\right)$ ✓ correct 3^{rd} root with correct angle. (no working required).

Question 5 **(7 marks)**

The complex number $z = 2 + i$ is a root of the polynomial equation $z^4 - 6z^3 + 16z^2 - 22z + q = 0$, where $q \in \mathbb{Z}$.

(a) State a second root of the equation. (1 mark)

$z = 2 - i$ ✓

(b) Determine the value of q and hence or otherwise, solve the equation $f(z) = z^4 - 6z^3 + 16z^2 - 22z + q = 0$. (5 marks)

$(z-2-i)(z-2+i) = z^2 - 2z + iz - 2z + 4 - 2i - iz + 2i + 1$

$= z^2 - 4z + 5$ ✓ obtains first quadratic factor.

$$
\begin{array}{r|l}
 & z^2 - 2z + 3 \\
\hline
z^2 - 4z + 5 & z^4 - 6z^3 + 16z^2 - 22z + q \\
 & -(z^4 - 4z^3 + 5z^2) \\
\hline
 & -2z^3 + 11z^2 - 22z \\
 & -(-2z^3 + 8z^2 - 10z) \\
\hline
 & 3z^2 - 12z + q \\
 & -(3z^2 - 12z + 15) \\
\hline
 & q - 15
\end{array}
$$

✓ obtains 2nd quadratic factor.

OR
$(z^2-4z+5)(z^2+bz+c) = z^4-6z^3+16z^2-22z+q$
$\therefore bz^3 - 4z^3 = -6z^3 \Rightarrow b = -2$
$-4cz - 10z = -22 \Rightarrow c = 3$
$(z^2-2z+3)(z^2-4z+5) = 0$
$\therefore q = 15$

$\therefore q - 15 = 0$ so $q = 15$ ✓ States 'q' value

$z^4 - 6z^3 + 16z^2 - 22z + 15 = 0$

$(z^2 - 4z + 5)(z^2 - 2z + 3) = 0$

$z^2 - 4z + 5 = 0$ and $z^2 - 2z + 3 = 0$

$z = 2 \pm i$ $\quad (z-1)^2 - 1 + 3 = 0$ ✓ solves for 2nd quadratic factor. Must consistently state $= 0$ to solve

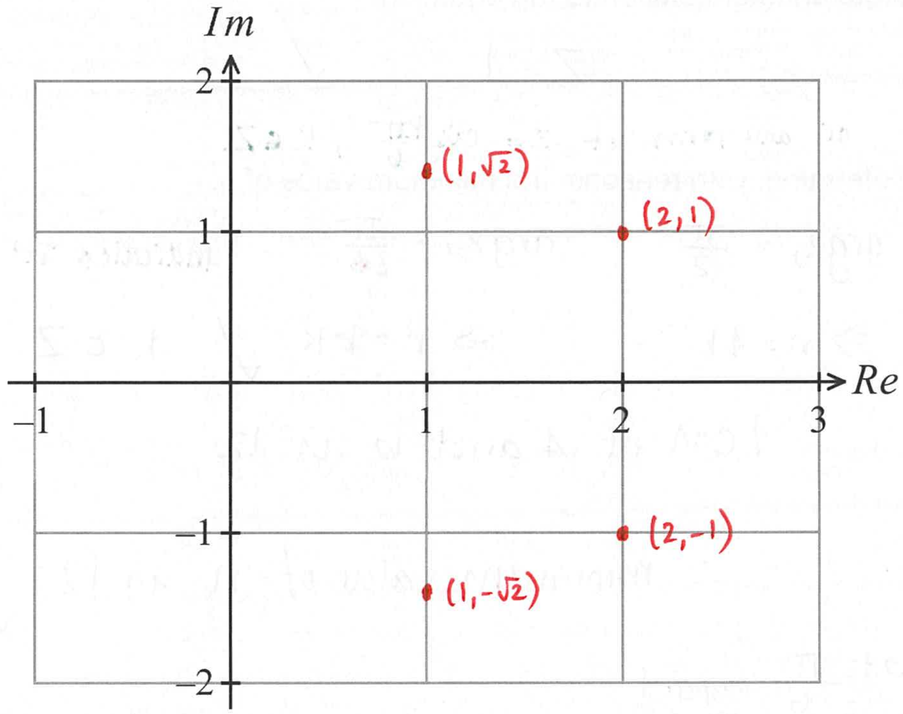
$z - 1 = \pm\sqrt{-2}$

$= \pm\sqrt{2}\,i$

$z = 1 \pm \sqrt{2}\,i$ ✓ States 3rd + 4th solution.

Question 5 continued

(c) Show the solutions to $z^4 - 6z^3 + 16z^2 - 22z + q = 0$ on an Argand diagram. (1 mark)



correct diagram.

Question 6 **(7 marks)**

(a) Two of the solutions to the equation $z^n = 1, n \in \mathbb{Z}^+$, are $\text{cis}\frac{\pi}{2}$ and $\text{cis}\frac{\pi}{3}$.

(i) State another solution to the equation. (1 mark)

$z = 1$ ✓

or any form of $z = \text{cis}\frac{k\pi}{6}$, $k \in \mathbb{Z}$

(ii) Determine, with reasons, the minimum value of n. (3 marks)

$\arg z_1 = \frac{\pi}{2}$ $\arg z_2 = \frac{\pi}{3}$ indicates 'n' is a multiple of 6

indicates 'n' is a multiple of 4.

✓ $\Rightarrow n = 4k$ $\Rightarrow n = 6k$ ✓ $k \in \mathbb{Z}$

LCM of 4 and 6 is 12

$\therefore$ Minimum value of n is 12. ✓ states min value of 'n'.

or each root $\frac{\pi}{6}$ apart

$\therefore$ 12 solutions in 2π.

(b) If $z = \text{cis}\left(\frac{\pi}{4}\right)$, determine the sum of the geometric series $1 + z_1 + z_2 + z_3 + \cdots + z_{24}$ and explain your answer. (3 marks)

$z = \text{cis}\left(\frac{\pi}{4}\right)$ this means 8 solutions ✓ determines 8 solutions for $z = \text{cis}\frac{\pi}{4}$

$|z| = 1$ so $z_0 + z_2 + \dots z_7 = 0$ ✓ indicates sum of first 8 terms $= 0$

can also be $1 + z_1 + \dots z_7 = 0$

and every 8 ~~consequ~~ consecutive terms will $= 0$

$\therefore$ $z_0 + z_1 + z_2 + \dots + z_{24}$

$= 0 + 0 + 0 + z_{24}$

$= \text{cis}\, 0$

$= 1$ ✓ determines sum

End of questions

Additional working space

Question number: _________

Additional working space

Question number: ___________